B.Sc Part I Paper I
(Hons)

Cardan's solution of a cubic equations

Q. Explain Cardan's method of solving the cubic equations

Ans:— let us suppose a cubic equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0 \quad \text{———} (1)$$

This is transformed into form

$$y^3+3Hy+G=0 \quad \text{———} (2)$$

Where $y=a_0x+a_1$, H & G have their usual meanings

Let $y=p^{\frac{1}{3}}+q^{\frac{1}{3}}$ is solution of (2)

$$\Rightarrow y^3 = p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}(p^{\frac{1}{3}}+q^{\frac{1}{3}})$$

$$\Rightarrow y^3 = p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}y$$

$$\Rightarrow y^3-3p^{\frac{1}{3}}q^{\frac{1}{3}}y-(p+q)=0 \quad \text{———} (3)$$

Comparing coefficient of identical terms of (2) and (3)

we get $-3p^{\frac{1}{3}}q^{\frac{1}{3}}=3H$

$$\Rightarrow H^3=-pq \text{ and } -(p+q)=G \Rightarrow G=-(p+q)$$

If p and q are the roots of the quadratic equation, then we get

$$z^2-(p+q)z+pq=0 \Rightarrow z^2+Gz-H^3=0$$

$$\Rightarrow z=\frac{-G\pm\sqrt{G^2+4H^3}}{2}$$

Let $p=\dfrac{-G+\sqrt{G^2+4H^3}}{2}$ and $q=\dfrac{-G-\sqrt{G^2+4H^3}}{2}$

Here when value of p and q are known then we can find the roots of equation (2) and so equation (3).

Further cube roots of p are $p^{\frac{1}{3}}$, $\omega p^{\frac{1}{3}}$, $\omega^2 p^{\frac{1}{3}}$ and that of q are $q^{\frac{1}{3}}$, $\omega q^{\frac{1}{3}}$, $\omega^2 q^{\frac{1}{3}}$ where ω is the cube root of unity. From these combinations of roots there be nine roots of the equation (2).

But these are restricted by the relation

$$p^{\frac{1}{3}} \cdot q^{\frac{1}{3}} = -H.$$

So, if $p^{\frac{1}{3}}$ take from first set then the corresponding element in the second set will be $q^{\frac{1}{3}}$. In the same way corresponding element of $\omega p^{\frac{1}{3}}$ will be $\omega^2 q^{\frac{1}{3}}$ and for $\omega^2 p^{\frac{1}{3}}$ will be $\omega q^{\frac{1}{3}}$.

So we have three and any three combinations which will satisfy

$$p^{\frac{1}{3}} q^{\frac{1}{3}} = -H$$

[viz. $(p^{\frac{1}{3}}, q^{\frac{1}{3}})$, $(\omega p^{\frac{1}{3}}, \omega^2 q^{\frac{1}{3}})$ and $(\omega^2 p^{\frac{1}{3}}, \omega q^{\frac{1}{3}})$]

Hence $y = p^{\frac{1}{3}} + q^{\frac{1}{3}}$, $\omega p^{\frac{1}{3}} + \omega^2 q^{\frac{1}{3}}$, $\omega^2 p^{\frac{1}{3}} + \omega q^{\frac{1}{3}}$ are three and only three values of y.

We have the value of x by the relations

$$y = a_0 x + a_1.$$

And so the complete solution of the cube is obtained.

Anjani Kumar Singh.